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A Method for Direct Structure Determinations in $P\bar{1}$ and Related Space groups

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In centrosymmetric symmorphic space groups the correct $\sum_2$ solution need not be one with the best consistency. In these space groups the correct $\sum_2$ solution can be found using a new criterion related to the Harker–Kasper inequalities:

$$\mathrm{HKC}=\sum_{\mathbf{h}}\sum_{\mathbf{k}}\frac{(|U_{\mathbf{h}}|-|U_{\mathbf{k}}|)^2}{(1-|U_{\mathbf{h}+\mathbf{k}}|)\,(1-|U_{\mathbf{h}-\mathbf{k}}|)}\,\mathrm{sign}\,(\mathbf{h}+\mathbf{k})\,\mathrm{sign}\,(\mathbf{h}-\mathbf{k})$$

in which the summations over **h** and **k** are restricted to those terms for which $|U_{\mathbf{h}}|$, $|U_{\mathbf{h}+\mathbf{k}}|$, $|U_{\mathbf{h}-\mathbf{k}}|$ are large and $|U_{\mathbf{k}}|$ is small. The weights in HKC are consistent with the Harker–Kasper inequalities for $\bar{1}$.

Introduction

In solving crystal structures by phase-relation methods it is generally expected that the most consistent solution based on

$$\varphi_{\mathbf{h}}=\frac{\sum_{\mathbf{k}}|E_{\mathbf{h}}E_{\mathbf{k}}E_{\mathbf{k}}|\,(\varphi_{\mathbf{k}}+\varphi_{\mathbf{h}-\mathbf{k}})}{\sum_{\mathbf{k}}|E_{\mathbf{h}}E_{\mathbf{k}}E_{\mathbf{h}-\mathbf{k}}|}$$

corresponds to the correct structure. However, in many space groups sets of trivial phases have better $\sum_2$ consistency (Schenk, 1972). In these cases a direct method, based on the $\sum_2$ relation only, cannot lead to the true structure.

In this paper a method is described for sign determinations in centrosymmetric symmorphic space groups, *i.e.* centrosymmetric space groups not containing glide planes or screw axes.

In these space groups the most consistent $\sum_2$ solution corresponds to the trivial set of signs $S(\mathbf{h})=+1$. Nevertheless the $\sum_2$ relation is very helpful, because for most of the high weights $|E_{\mathbf{h}}E_{\mathbf{k}}E_{\mathbf{h}-\mathbf{k}}|$ the sign relations $S(\mathbf{h})\,S(\mathbf{k})\,S(\mathbf{h}-\mathbf{k})=+1$ hold true. Thus by using strict acceptance criteria a set of signs and symbolic signs may be built up, which contains the correct solution. On the other hand the true values of the symbols cannot be determined by means of a $\sum_2$ consistency criterion, because if symbol $i=$ symbol j the contribution to the consistency criterion is always better than if symbol $i=-$symbol j. From this it follows that although one of the $\sum_2$ solutions may correspond to the correct structure, this need not be one with good $\sum_2$ consistency.

Harker–Kasper inequalities for discriminating between Σ_2 solutions

In order to discriminate the correct $\sum_2$ solution in centrosymmetric symmorphic space groups the only useful sign relations are those which lead to negative signs. This may be achieved by the Harker–Kasper inequalities for $\bar{1}$:

$$(U_{\mathbf{h}}+U_{\mathbf{k}})^2\le(1+U_{\mathbf{h}+\mathbf{k}})\,(1+U_{\mathbf{h}+\mathbf{k}})$$
$$(U_{\mathbf{h}}-U_{\mathbf{k}})^2\le(1-U_{\mathbf{h}+\mathbf{k}})\,(1-U_{\mathbf{h}-\mathbf{k}})\,.$$

If $|U_{\mathbf{h}}|$, $|U_{\mathbf{h}+\mathbf{k}}|$ and $|U_{\mathbf{h}-\mathbf{k}}|$ are large, $U_{\mathbf{k}}=0$ and

$$U_{\mathbf{h}}^2>(1-|U_{\mathbf{h}+\mathbf{k}}|)\,(1-|U_{\mathbf{h}-\mathbf{k}}|) \qquad (1)$$

it follows that the sign relation

$$S(\mathbf{h}+\mathbf{k})S(\mathbf{h}-\mathbf{k})=-1 \qquad (2)$$

must hold (see *e.g.* Woolfson, 1961). In case reflexions $\mathbf{h}+\mathbf{k}$ and $\mathbf{h}-\mathbf{k}$ both have a symbolic sign, determined by means of the $\sum_2$ relation, expression (2) gives a symbol product

$$\text{symbol } i.\ \text{symbol } j\ldots\ \text{symbol } n=-1\,.$$

In this way it is possible to break the deadlock.

This approach has been applied in the determination of the unknown structure of a cyclopropane derivative of space group $P\bar{1}$ with $Z=2$ and $N=20$. By means of the $\sum_2$ relation, approximately 100 strong re-

Table 1. *Classification of the* 8 $\sum_2$ *solutions of a cyclopropane derivative* (*space group* $P\bar{1}$) *on the basis of the* $\sum_2$ *consistency criterion and the HKC values*

The smallest value of both criteria should correspond to the true structure. The reflexions **h**, $\mathbf{h}-\mathbf{k}$ and $\mathbf{h}+\mathbf{k}$ belong to the 150 strongest reflexions and $|U_{\mathbf{k}}|<1$.

$\sum_2$ solution	$\sum_2$ consistency criterion	Order of solutions after CC	HKC	Order of solutions after HKC
1	0	1	21·68	8
2	0	1	11·64	7
3	108	5	−13·53	2
4	108	5	−16·15	1
5	0	1	− 3·72	4
6	0	1	6·32	6
7	108	5	− 4·43	3
8	108	5	− 1·82	5

Table 2. $\sum_2$ *consistency criteria and HKC values for the structure of Vitamin A acid* (*space group* $P\bar{1}$, $Z=2$, $N=44$)

	Criteria on the basis of 151 signed reflexions				Criterion on the basis of 194 signed reflexions	
Number of the $\sum_2$ solution	$\sum_2$ consistency criterion CC	Order of solutions after CC	HKC	Order of solutions after HKC	HKC	Order of solution after HKC
1	0	1	25·21	32		
2	129	11	11·63	30		
3	105	7	7·42	26		
4	226	23	8·38	27		
5	200	19	−0·80	11	− 4·21	10
6	314	31	−0·61	13	− 7·66	4
7	162	13	−4·66	4	− 7·34	5
8	279	27	−6·98	3	−11·68	1
9	181	15	−0·93	10	− 1·58	12
10	293	29	1·86	19		
11	94	5	−2·11	8	− 4·72	9
12	215	21	−3·07	6	− 5·21	7
13	68	3	4·91	22		
14	195	17	3·38	21		
15	123	9	0·00	15		
16	247	25	−2·31	7	− 0·48	13
17	68	4	18·96	31		
18	195	18	5·38	23		
19	123	10	6·11	24		
20	247	26	7·06	25		
21	181	16	1·24	17		
22	293	30	1·44	18		
23	94	6	0·86	16		
24	215	22	−1·45	9	− 5·07	8
25	200	20	−3·54	5	− 6·10	6
26	314	32	−0·74	12	− 3·58	11
27	162	14	−7·07	2	−10·04	3
28	279	28	−8·03	1	−10·99	2
29	0	2	10·59	29		
30	129	12	9·06	28		
31	105	8	1·89	20		
32	226	24	−0·43	14	3·05	14

Table 3. $\sum_2$ *consistency criteria and HKC values for the structure of an aza-steroid* (*space group* $P\bar{1}$, $Z=2$, $N=40$)

Out of 32 possible $\sum_2$ solutions only the values of the solutions with lowest HKC are given.

	Criteria on the basis of 205 signed reflexions				Criterion on the basis of 275 signed reflexions	
Number of the $\sum_2$ solution	$\sum_2$ consistency criterion CC	Order of solutions after CC	HKC	Order of solutions after HKC	HKC	Order of solutions after HKC
2	1267	12	−28·69	1	−40·17	1
5	4335	27	−21·84	3	−29·41	2
9	1525	15	−19·75	4	−26·01	5
12	1158	8	−6·28	13	−9·64	11
15	4601	31	−7·24	11	−9·97	10
16	4154	20	−4·63	14	−1·90	14
18	1212	10	−11·10	9	−9·07	12
20	1525	16	−16·71	6	−18·25	7
21	4203	23	−6·61	12	−2·56	13
23	4614	32	−23·15	2	−28·47	3
26	878	7	−9·86	10	−13·92	9
27	1506	14	−14·60	8	−16·92	8
29	4344	28	−16·30	7	−19·19	6
32	4240	25	−17·23	5	−26·12	4

flexions were signed using three symbols. The $\sum_2$ consistency criteria of the 8 solutions are given in Table 1. From a small number of sign relations (2) it could be concluded that either solution 3 or solution 4 is the correct one, in spite of their bad $\sum_2$ consistency. Solution 4 revealed the complete structure.

The method outlined above is more or less similar to that used by Woolfson (1961) in his Chapter 2. The difference is that for the initial phasing process the $\sum_2$ relation is used where Woolfson employed Harker–Kasper inequalities.

However, both methods are subject to the limitations of the Harker–Kasper inequalities. In various textbooks it is pointed out that only structures up to $N=16$–18 can be solved by means of inequalities, so that it is clear that the cyclopropane derivative is one of the largest structures which could be solved in this way.

Extension of the method

In our improved method the relations (2) are used in the following way: If $|U_{\mathbf{h}}|$, $|U_{\mathbf{h}+\mathbf{k}}|$, $|U_{\mathbf{h}-\mathbf{k}}|$ are large and $|U_{\mathbf{k}}|$ is small then the reliability of a relation (2):

$$S(\mathbf{h}+\mathbf{k})S(\mathbf{h}-\mathbf{k})=-1$$

is taken to be proportional to:

$$W_{\mathbf{hk}}=\frac{(|U_{\mathbf{h}}|-|U_{\mathbf{k}}|)^2}{(1-|U_{\mathbf{h}+\mathbf{k}}|)\,(1-|U_{\mathbf{h}-\mathbf{k}}|)} \qquad (4)$$

which is based on (1). Then for each $\sum_2$ solution the following Harker–Kasper criterion (HKC) can be calculated:

$$\mathrm{HKC}=\sum_{\mathbf{h}}\sum_{\mathbf{k}}\frac{(|U_{\mathbf{h}}|-|U_{\mathbf{k}}|)^2\,S(\mathbf{h}+\mathbf{k})S(\mathbf{h}-\mathbf{k})}{(1-|U_{\mathbf{h}+\mathbf{k}}|)\,(1-|U_{\mathbf{h}-\mathbf{k}}|)}. \qquad (5)$$

The $\sum_2$ solution with the smallest HKC is expected to be the correct solution.

The HKC values have been calculated for the 8 $\sum_2$ solutions of the cyclopropane derivative and are given in Table 1. Whereas by means of the pure inequalities (2) two different solutions (3 and 4) were obtained, for the correct solution 4 the HKC value is smallest.

Practical procedure for structure determinations

We have adopted the following procedure in our computer programs.

(1) Calculation of $\sum_2$ list and production of the doublet sign relations (2) with their weights (4).

(2) Construction by means of the $\sum_2$ relation of a set of symbolically signed structure factors, which contains as many of the strong reflexions as possible.

(3) Calculation of the HKC for each solution.

(4) Extension of the group of signed reflexions for the $\sum_2$ solutions with the best HKC values. Back to step 3.

Two practical points are very important.

(1) The method depends critically on the small U values.

(2) The weights (4) are very sensitive to errors in the scaling factor q, as can be seen from

$$W_{\mathbf{hk}}=\frac{q^2(|U_{\mathbf{h}}|^2-|U_{\mathbf{k}}|^2)}{(1-q|U_{\mathbf{h}+\mathbf{k}}|)\,(1-q|U_{\mathbf{h}-\mathbf{k}}|)}.$$

Applications

Up till now the HK criterion has been tested in four structure determinations and in all four cases the method was successful. Two of the structures had previously been solved by means of the Patterson method; the two others were unknown. The structure determination of one of the unknown structures has already been described in the preceeding sections.

The second structure tackled was Vitamin A acid (Stam, 1972), space group $P\bar{1}$, $Z=2$, $N=44$. In the second step 151 reflexions out of the 300 strongest were signed on the basis of 5 symbols. The $\sum_2$ consistency of all 32 solutions is given in Table 2, together with the HKC values. For the solutions with a negative HKC the set of signed structure factors was extended to 194. New HKC values were calculated and are given in Table 2. The solution with the lowest HKC value proved to be correct; this is a solution with one of the worst $\sum_2$ consistency values.

The next application of the method was the structure determination of an unknown aza-steroid (space group $P\bar{1}$, $Z=2$, $N=40$). In the last two years different coworkers of our laboratory tried to solve this structure by means of direct and Patterson methods without success. Using 5 symbols, 205 reflexions were signed. On the basis of the HKC values 14 solutions were selected (see Table 3) and for these solutions the number of signed reflexions was extended to 275. An E map of the solution with the smallest HKC value revealed the complete structure. This solution was number 12 in order of increasing $\sum_2$ consistency.

Our practical procedure has also been applied to the structure of cyclopropane-1,1-dicarboxylic acid, space group $P\bar{1}$, $Z=4$, $N=36$ (Meester, Schenk & MacGillavry, 1971). The hypercentric intensity statistics indicated the presence of a pseudo centre between the two crystallographically independent molecules. The results of the application of our practical procedure to this structure using 6 symbols has been summarized in Table 4. Out of the 64 solutions number 21 has the best HKC value (−47·11) but its E map did not show the structure. The next best solutions in HKC are numbers 17, 8 and 18 (HKC values −39·13, −38·26 and 37·34 respectively), but the differences between them are very small. From the improvement of the HKC values of these solutions in the three cycles (−16, −16 and −22 respectively, see Table 4) it can be expected that after a further extension of the set of signed reflexions, solution number 18 will get the second-best

Table 4. $\sum_2$ *criteria and HKC values for cyclopropane-1,1-dicarboxylic acid* ($P\bar{1}$, $Z=4$, $N=36$)

Out of 64 different $\sum_2$ solutions the 20 best HKC solutions are given.

	Cycle 1 Criteria on the basis of 210 signed reflexions				Cycle 2 Criterion on the basis of 264 signed reflexions		Cycle 3 Criterion on the basis of 328 signed reflexions	
Number of the $\sum_2$ solution	$\sum_2$ consistency criterion CC	Order of solutions after CC	HKC	Order of solutions after HKC	HKC	Order of solutions after HKC	HKC	Order of solutions after HKC
1	1702	29	−15·89	11	−23·60	9	−29·64	7
2	2975	56	−11·11	18	−11·86	20		
4	3112	60	−20·64	4	−29·01	4	−29·72	6
5	1892	36	−16·63	8	−23·61	8	−28·27	9
6	815	7	−12·27	17	−15·43	18		
7	1837	32	−8·51	20	−21·55	11	−29·00	8
8	910	8	−22·07	3	−34·81	2	−38·26	3
17	2505	47	−22·93	2	−34·48	3	−39·13	2
18	1320	15	−15·39	12	−27·63	5	−37·34	4
19	2464	45	−16·37	9	−20·99	12	−24·16	12
20	1448	20	−13·52	15	−18·02	14		
21	2035	38	−28·69	1	−41·55	1	−47·11	1
22	3280	62	−13·54	14	−25·18	6	−26·37	10
23	1955	37	−17·93	6	−24·63	7	−29·93	5
24	3352	63	−16·89	7	−22·55	10	−25·49	11
33	1695	28	−16·25	10	−16·12	16		
35	1629	25	−10·47	19	−12·95	19		
37	1890	35	−18·36	5	−19·60	13		
49	2705	52	−13·06	16	−15·59	17		
53	2164	40	−14·73	13	−17·62	15		

HKC value. An E map of this solution contains the complete structure. This solution is number 15 in order of increasing $\sum_2$ consistency. It must be noted here that in our experience pseudo symmetry always gives rise to a more difficult direct phase determination, so that it was not surprising that the last structure determination gave more trouble than the three others.

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